Tutorial 2 2022/9/28

2.1 Improper multiple integral[1]

Improper integral is applied when we want to integrate the function over some unbounded domain or integrate some unbounded function. The idea behind it is to use something finite to approach something infinite, and likewise use bounded subsets to approach unbounded subsets.

Let's begin with the definition for improper multiple integral, but firstly we need to define some notions from topology to make the definition coherent and rigorous.

Definition 2.1

Let E be a subset of $\mathbb{R}^m$, E is bounded if there is a number R such that for all $x = (x_1, \cdots, x_m) \in E$ we have $\sum_{i=1}^m x_i^2 \leq R$.

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Definition 2.2

The closure of $E \subset \mathbb{R}^m$ is the subset $\{x \in \mathbb{R}^m | \exists x_n \in E, \lim_{n\to\infty} x_n = x\}$, denoted by $\bar{E}$.

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Definition 2.3

The boundary ∂E of E is the intersection of closure of E and the closure of the complement of E. That is, $\partial E = \bar{E} \cap \bar{E}^c$.

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Example 2.1 The closure of an open interval (a, b) is the closed interval $[a, b]$. The boundary of (a, b) is the set of the two endpoints $\{a, b\}$

In the following we write $\mathrm{d}\, x_1 \,\mathrm{d}\, x_2 \cdots \mathrm{d}\, x_m$ as $\mathrm{d}\, x$ for short.

Definition 2.4

A subset $E \subset \mathbb{R}^m$ is measurable if E is bounded, the characteristic function χ_E is integrable, and $\chi_{\partial E}$ is integrable and has integral $\int \chi_{\partial E} \,\mathrm{d}\, x = 0$.

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Definition 2.5

An exhaustion of a set $E \subset R^m$ is a sequence of measurable subsets E_n such that $E_n \subset E_{n+1} \subset E$ for any $n \in \mathbb{N}$ and $\bigcup_{n=1}^\infty E_n = E$.

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Example 2.2 $E_n := [-n, n]^2$ is a exhaustion of $\mathbb{R}^2$.

Definition 2.6

Let $\{E_n\}$ be an exhaustion of the set E and suppose the function $f : E \to \mathbb{R}$ is integrable on the sets $E_n \in \{E_n\}$. If the limit

$$\int_E f(x)\mathrm{d}x := \lim_{n\to\infty} \int_{E_n} f(x)\mathrm{d}x$$

exists and has a value independent of the choice of the sets in the exhaustion of E, this limit is called the improper integral of f over E.

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[1] For the reference of this section, I copied the chapter 11.6 of the book: Zorich, Vladimir Antonovich, and Octavio Paniagua. Mathematical analysis II. Vol. 220. Berlin: Springer, 2016.

We do need to check the independence for all exhaustions. Consider $f(x) = \sin x$. Then $\int_0^{2n\pi} f(x)\,\mathrm{d}\,x = 0$ but $\int_0^{(2n+1)\pi} f(x)\,\mathrm{d}\,x = 2$. So we may have two integral for $\sin x$ if we consider only one exhaustion, which does not make sense.

For non-negative functions we don't need to check for all exhaustions.

Proposition 2.1

If a function $f : E \to \mathbb{R}$ is nonnegative and the limit in Definition 2.6 exists for even one exhaustion $\{E_n\}$ of the set E, then the improper integral of f over E converges.

Example 2.3 Let us find the improper integral $\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\mathrm{d}x\,\mathrm{d}y$.

We shall exhaust the plane $\mathbb{R}^2$ by the sequence of disks $E_n = \left\{(x,y) \in \mathbb{R}^2 \mid\ x^2+y^2 < n^2\right\}$. After passing to polar coordinates we find easily that

$$\iint_{E_n} \mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi}\mathrm{d}\varphi\int_0^n \mathrm{e}^{-r^2}\,\mathrm{d}r = \pi\left(1-\mathrm{e}^{-n^2}\right) \to \pi, \quad \text{as } n \to \infty.$$

By Proposition 2.1 we can now conclude that this integral converges and equals π. One can derive a useful corollary from this result if we now consider the exhaustion of the plane by the squares $E_n' = \left\{(x,y) \in \mathbb{R}^2 || x| \le n \wedge |y| \le n\right\}$.

$$\iint_{E_n'} \mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\,\mathrm{d}y = \int_{-n}^{n}\mathrm{d}y\int_{-n}^{n} \mathrm{e}^{-(x^2+y^2)}\mathrm{d}x = \left(\int_{-n}^{n}\mathrm{e}^{-t^2}\,\mathrm{d}t\right)^2$$

By Proposition 2.1 this last quantity must tend to π as $n \to \infty$. Thus, following Euler and Poisson, we find that

$$\int_{-\infty}^{+\infty}\mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$$

This is the so-called Gaussian integral, often used in statistics and physics.

2.2 Characteristic function

Let's look at another problem from the assignment:

Problem 2.1 Let S be a non-empty set in $\mathbb{R}^n$. Define its characteristic function χ_S to be $\chi_S(\mathbf{x}) = 1$ for $\mathbf{x} \in S$ and $\chi_S(\mathbf{x}) = 0$ otherwise. Prove the following identities: (a) $\chi_{A\cap B} = \chi_A\chi_B$. (b) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$.

We can prove for more general cases. Let X, Y be any set. Denote the set of all maps from X to Y by Y^X. Here we consider $Y = \{0,1\}$. Then the elements in $\{0,1\}^X$ are called the characteristic functions on X. Any such function is defined in the same way as χ_S for some $S \subset X$.

To prove two functions are equal, it is to prove their values at each point are equal.

We first consider the equation $\chi_{A\cap B} = \chi_A\chi_B$. By definition $\chi_{A\cap B}(x) = 1$ if and only if $x \in A\cap B$. And $\chi_A(x)\chi_B(x) = 1$ if and only if $\chi_A(x) = \chi_B(x) = 1$, if and only if $x \in A$ and $x \in B$, which is equivalent to $x \in A \cap B$. Therefore we have $\chi_{A\cap B} = \chi_A\chi_B$.

For (b) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$ we can argue in a similar way by checking when the both sides equal to 1. There is another method to prove it and I would like to show that for general cases. Firstly we assume the following formula for the indeterminants $a_1, \cdots, a_n$.

Lemma 2.1

$$(1-a_1)(1-a_2)\times\cdots\times(1-a_n) = \sum_{k=1}^{n}\prod_{1\le i_1<\cdots<i_k\le n}(-1)^k a_{i_1}a_{i_2}\cdots a_{i_k}$$

Proposition 2.2

Let $S_1, S_2, \cdots, S_n$ be subsets of X, then we have

$$\chi_{S_1\cup\cdots\cup S_n} = \sum_{k=1}^{n} \prod_{1\le i_1<\cdots<i_k\le n} (-1)^{k-1} \chi_{S_{i_1}\cap S_{i_2}\cap\cdots\cap S_{i_k}}$$

Proof By substituting S_i into a_i in lemma 2.1, we have

$$(\chi_X - \chi_{S_1})(\chi_X - \chi_{S_2}) \times \cdots \times (\chi_X - \chi_{S_n}) = \sum_{k=1}^{n} \prod_{1\le i_1<\cdots<i_k\le n} (-1)^k \chi_{S_{i_1}} \chi_{S_{i_2}} \cdots \chi_{S_{i_k}}$$

By induction on (a) we know that $\chi_{S_{i_1}} \chi_{S_{i_2}} \cdots \chi_{S_{i_k}} = \chi_{S_{i_1}\cap S_{i_2}\cap\cdots\cap S_{i_k}}$.

Since $\chi_S + \chi_{S^c} = \chi_X$, for the left part we have

$$\begin{aligned}(\chi_X - \chi_{S_1})(\chi_X - \chi_{S_2}) \times \cdots \times (\chi_X - \chi_{S_n}) &= \chi_{S_1^c}\chi_{S_2^c}\cdots\chi_{S_n^c} \\ &= \chi_{S_1^c\cap\cdots\cap S_n^c} \\ &= \chi_{(S_1\cup\cdots\cup S_n)^c} \\ &= \chi_X - \chi_{S_1\cup\cdots\cup S_n}\end{aligned}$$

Substract χ_X from both sides we proved the proposition.

Remark This is a kind of the so-called inclusion exlusion principle.

Remark For more generalizations, investigate the term "Boolean ring", "fuzzy set".

2.3 Volume of tetrahedron[2]

Now we switch to another problem. We know that the area of a triangle is $\frac{1}{2}a \times h$ where a is the length of the base of the triangle and h is the height to the base. While for a tetrahedron we know its volume is $\frac{1}{3}S \times h$ where S is the area of the base of the tetrahedron and h is the height to the base.

One might imagine how those people living in a world of dimension 4 calculate the "volume" of a "tetrahedron" of dimension 4 and one may guess the formula $\frac{1}{4}S \times h$ still holds. And generally, aliens in $\mathbb{R}^n$ should have the formula $\frac{1}{n}S \times h$.

To check this, one could first assume that the formula hold for all "tetrahedrons" if and only if it hold for a standard "tetrahedron". We define the standard "tetrahedron" in $\mathbb{R}^m$ to be the subset $\Delta_m := \{(x_1, \cdots, x_m) \in \mathbb{R}^m | 0 \le x_i \le 1, \forall 1 \le i \le m, x_1 + \cdots + x_m \le 1\}$, we call Δ_m the standard simplex.

The area of Δ_2 is

$$\int_0^1 \int_0^{1-x_2} 1 \,\mathrm{d}x_1 \,\mathrm{d}x_2 = \frac{1}{2}$$

The volume of Δ_3 is

$$\int_0^1 \int_0^{1-x_3} \int_0^{1-x_2-x_3} 1 \,\mathrm{d}x_1 \,\mathrm{d}x_2 = \frac{1}{2} \times \frac{1}{3}$$

Analogously you will find the volume of Δ_m is

$$\int_0^1 \int_0^{1-x_m} \cdots \int_0^{1-x_3-\cdots-x_m} \int_0^{1-x_2-x_3-\cdots-x_m} 1 \,\mathrm{d}x_1 \,\mathrm{d}x_2 \cdots \mathrm{d}x_m = \frac{1}{2} \times \frac{1}{3} \times \cdots \times \frac{1}{m} = \frac{1}{m!}$$

So we do find that the formula $\frac{1}{m}S \times h$ hold for the volume of "tetrahedron" in dimension m.

Leave it behind, now consider the following experiment: Let's pick elements from $[0, 1]$ randomly, until

[2]The reference for this section is https://zhuanlan.zhihu.com/p/369714158

their sum gets larger than 1. Then we record the number of elements we have chosen. For example, if we got two random numbers 0.12, 0.57 at first and we got 0.41 for the third pick, then we record $n_1 = 3$ as now $0.12 + 0.57 + 0.41 > 1$. Now we do this step repeatedly and we get a sequence of number $n_1, n_2, \cdots, n_k, \cdots$. We call $A = \lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} n_i$ the average of number of tries that we need to pick the numbers until their sum is larger than 1. Now we try to calculate it.

Let $\Pr[Y = i]$ be the probability that we need to pick exactly i numbers from $[0, 1]$ so that their sum exceeds 1. A probability theorem will tell you that

$$A = \sum_{i=1}^{\infty} i \cdot \Pr[Y = i]$$

We can rewrite it in the following way

$$\begin{aligned}
\sum_{i=1}^{\infty} i \cdot \Pr[Y = i] \quad &= 1 \cdot \Pr[Y = 1] + 2 \cdot \Pr[Y = 2] + 3 \cdot \Pr[Y = 3] + \dots \\
&= \sum_{i=1}^{\infty} \Pr[Y = i] + (1 \cdot \Pr[Y = 2] + 2 \cdot \Pr[Y = 3] + \dots) \\
&= \sum_{i=1}^{\infty} \Pr[Y = i] + \sum_{i=2}^{\infty} \Pr[Y = i] + (1 \cdot \Pr[Y = 3] + 2 \cdot \Pr[Y = 4] + \dots) \\
&= \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \Pr[Y = i] \\
&= \sum_{k=1}^{\infty} \Pr[Y \geq i]
\end{aligned}$$

If we choose i times from $[0, 1]$, then we have the possible numbers $a_1, a_2, \cdots, a_i$. Then $\Pr[Y \geq i]$ is the probability that $a_1 + a_2 + \cdots + a_{i-1} \leq 1$, so it is exactly the quotient of the volume of Δ_{i-1} by the volume of $[0, 1]^{i-1}$, which is $\frac{1}{(i-1)!}$. Therefore, the avarage of number of tries $A = \sum_{i=1}^{\infty} \frac{1}{(i-1)!} = e$.

Remark The above procedure is a way to approximate e using Monte Carlo Simulation.

Exercise 2.1 Try to find the average of number of tries that we need to pick numbers from $[0, 1]$ randomly until their sum exceeds $x \in [0, 1]$.